EXERCISES FOR CHAPTER FIVE

1. Let X be a random variable with probability density function given by: $f_X(x) = 1/5 \quad (0 < x < 5)$.
   a. Identify the theoretical distribution and find the mean and the variance of X.
   b. Consider two independent random variables X_1 and X_2 with common p.d.f. . Show that the r. v. $Y = X_1 + X_2$ doesn't belong to the family of the distributions of the parcels.
   c. Let $Y = 1/X$, compute its mean if it exists.

2. Let X be a random variable with a uniform distribution in the interval $(-\theta, \theta)$ $\theta > 0$. Find the value of θ so that $P(X > 4) = 0.25$.

3. Show that if $X \sim U(0,1)$, then $Y = a + (b - a)X \sim U(a, b)$ $a < b$.
4. Let Z be a random variable with a standard normal distribution. Compute, using the graphical representation where appropriate:
   a. $P(0 < Z \leq 2.05)$; b. $P(-1.22 < Z \leq 1.05)$; c. $P(Z \geq -2.05)$;

   d. The value k such that $P(|Z| > k) = 0.05$.

   e. The value k such that $P(|Z| < k) = 0.90$.

5. Let X be a random variable with a normal distribution with mean 6 and variance 25. Compute:

   a. $P(6 < X \leq 12)$; b. $P(0 < X \leq 8)$; c. $P(X < 4)$;

   d. $P(|X - 6| > 10)$; e) The value k such that $P(X > k) = 0.90$.

6. What is the distribution of a random variable X with the following m.g.f.: $M_X(s) = e^{2s+s^2}$?

7. The daily amount of deposits made daily at a certain bank, follows a normal distribution with mean 120 monetary units and variance 64.

   a. Determine the percentage of days when the amount of deposits lies between 105 and 135 monetary units .

   b. Compute the probability that the amount of deposits be above average in the days when that amount is less than 125 monetary units.

   c. Calculate the mean and variance of the weekly amount of deposits (5 days).

   d. Find the lower limit of the amount of deposits which occurs in 90% of days.

8. The time spent in a book fair is a random variable with normal distribution with mean 2 hours. Assume that only 2.5% of visitors stay more than 3 hours.

   a. What is the standard deviation of the random variable?

   b. Knowing that a visitor has arrived an hour ago, how likely it is that he leaves the fair in the next 30 minutes?

   c. Calculate the median and interquartile range of X and interpret its meaning.

   d. Calculate the probability that in 20 visitors, randomly chosen, at most one remains more than 3 hours.

9. For marketing purposes, certain fruits are sorted by size in cms, taking as measure its maximum diameter. The maximum diameter of the fruits is a normally distributed random variable with variance equal to 5 and mean μ. The categories are as follows: C1-fruit with a maximum diameter less than or equal to 6; C2 - fruit with a maximum diameter between 6 and 12; C3 - fruit with a maximum diameter greater than or equal to 12.

   a. Knowing that 30% of the fruits are in category C3, compute the average maximum diameter of the fruits and the percentage of fruit in each of the other categories.

   b. If the fruits are sold in packs of 6 units, including all sizes randomly, what is the likelihood that in a pack, at least 2 fruits are of category C3.

10. Axles produced by a certain machine are not considered defective if the deviation of the diameter of the axle relative to the norm is not larger, in absolute value, than 2 mm. The random deviations of the diameter of the axles follow a normal distribution of zero mean and standard deviation of 1.6mm. Determine the percentage of non defective axles produced.

11. At an establishment that sells building materials, it is known that daily sales of sand (in hundreds of kilograms) has a random behavior, translated by a normal distribution with mean 20 and standard deviation 2.

   a. Knowing that one morning the establishment has sold a ton of sand, what is the probability that in the same day the amount of sand sold is more than 2.5 tons?

   b. What is the probability that, in any given month (20 days), sales exceed 37 tons of sand?

12. Let X be a random variable with normal distribution with mean 100 and variance 225, which represents the results for a person in a psychometric test.

a. What percentage of people score between 80 and 115?

b. Knowing that the result was higher than the median, find the probability that it is less than 115?

c. From 20 people selected at random, what is the probability that at least half have a result above the 3rd quartile?

d. Compute $P[(X-100)^2 \leq 144]$.

13. Let X be a random variable with exponential distribution with mean 20. Compute:

a. $P(10 < X \leq 30)$; $b. P(X > 30)$; c. $P(X > 40|X > 10)$.

14. A bank serves, on average, two customers every 3 minutes. Consider that customers are served according to a Poisson process.

a. How likely is 3 minutes elapse without any client served?

b. What is the likelihood that the customer takes more than 3 minutes to be served?

c. Compare the results of the previous paragraphs.

d. What is the likelihood of a client having to wait between 3 and 6 minutes to be served?

15. In a particular airport, aircrafts land at a rate of 2 per hour, according to a Poisson process.

a. How likely is that the time between two consecutive arrivals is less than 15 minutes?

b. How likely is that the waiting until the next arrival be over half an hour?

16. Acquired up a box with 12 light bulbs which provided an indication that the average duration of life of a bulb is 1000 hours. Assuming that the lifetime of a bulb follows an exponential distribution, determine the probability of the bulb that has the shortest duration, between the 12 in the box , lasts more than 50 hours.

17. In an industrial unit, the runtime of a part is a random variable with exponential distribution with a mean of 5 minutes.

a. It is known that one part is already running for 2 minutes. Determine the probability that at least 4 minutes are still required to completion. Comment.

b. Five pieces are selected at random, calculate the probability that two of them have had a maximum run time of 4 minutes.

c. Assuming that there are no parts in stock, think it reasonable that, at some point, the company has committed to provide 50 pieces within 4 hours? Justify.

18. John has 4 girl friends which always arrive late to their meetings. Assume that the delay times of each one of them are independent and have an exponential distribution with mean 20 minutes. John is never late.

a. John invited one of them for lunch but decided that this time would not wait more than 30 minutes. How likely John's having lunch with the invited friend that day?

b. If John invite all 4 for dinner what is the probability of having to wait more than 10 minutes until the first appear? How likely is it that John has to wait more than 1 hour until the last to appear?

19. The time elapsed since failure until repair (designated as repair time) of a certain type of machines is a random variable with exponential distribution with mean of 2 hours.

a. What is the probability that a broken machine has a repair time of 1 hour?

b. If 10 broken machines were randomly selected, what is the probability of the fastest repair be performed in less than 15 minutes?

c. What is the probability that the total repair time of 50 broken machines does not exceed 90 hours?

20. Using the moment generating function, show that:

$$X \sim G(\alpha, \lambda) \text{ and } Y = X/c\,,\ \ c > 0\ \text{ then } Y \sim G(\alpha, c\lambda)$$

21. Let X be a random variable following a Chi-square distribution with 23 degrees of freedom, find:

a. $P(14.85 < X \leq 32.01)$;

b. a and b such that $P(a < X < b) = 0.95$ and $P(X < a) = 0.025$.

c. The mean and variance of X.

d. $\chi^2_{23,0.05}$ and $\chi^2_{23,0.95}$.

22. Let $X \sim G(5; 2)$. Compute using the chi-square distribution:

a. $P(X > \mu_X)$.

b. Value k such that $P(X > k) = 0.05$.

23. Let X and Y be independent random variables such that $X \sim Ex(4)$ and $Y \sim \chi^2_{(5)}$. $W = 8X + Y$.

a. Find the mean and standard deviation of W.

b. Find value k such that $P(W < k) = 0.75$.

24. Using the moment generating function, show that the square of a standard normal distribution follows a chi-square distribution with 1 degree of freedom.

25. Assume that the links of a bicycle chain have random lengths with mean 0.5 cm and standard deviation 0.04 cm. The standards for a bicycle manufacturer require that the length of a chain is between 49 and 50 cm.

a. If a chain has 100 links, determine the percentage of chains that meet the standards required.

b. If the chain has only 99 links, what should be the standard deviation of the population so that 90% of the chains meet the standards of the manufacturer?

26. In a box of a grocery, bills that customers have to pay are always rounded to the nearest multiple of 5. Thus, for each account, the difference between the received and registered may be considered a random variable with probability function given by: $f_X(x) = \frac{1}{5} \quad (x = -2, -1, 0, 1, 2)$.

On a day when 100 clients are served, what is the probability that the difference between the total amount received and recorded exceeds 40 cents?

27. A factory produces article A at the rate of 100 units per day. The amount of raw material B embedded in each article is a random variable with mean 75 and variance 225. Determine the percentage of days the raw material consumption does not exceed 7.6 kg. Items are sold in lots of 200. Assuming the cost of raw materials is 0.25 euros / gram, what should be the sales price of the lot, to cover the cost of the raw material in 95% of situations?

28. A square of Lisbon usually has cars parked in contravention. The Police fines, every day, 90% of cars parked in violation, leaving the notification in windshield. The number of people who present themselves at the police station to pay fines, per day, is a random variable with mean and variance equal to 10. If each fine is 20 euros and the police station is open 225 days a year, what is the probability that the total amount of fines collected annually exceed 47,000 euros?

29. A factory knows that minimizes its cost per unit if a daily production of at least 300 units is achieved. Admit be 0.3 the probability that this level of production is not reached.

a. Considering 100 days of operation, what is the probability that such a level is not reached at more than 20 and up to 40 days?

b. If n = 20 and the random variable Y represents the number of days during which the production is less than 300 units, compare the accurate and approximate probabilities of the event $12 \leq Y \leq 16$.

30.Consider $X \sim B(100; 0.1)$. Compute $P(12 \leq Y \leq 14)$ using:

a. The normal distribution with the correction for continuity.

b. The Law of rare events (Poisson distribution).

c. The Binomial distribution.

31. In a particular gloves factory, 1% of the gloves produced have defects, so that the number of defective gloves in each pair follows a binomial distribution $X \sim B(2; 0.01)$. Calculate the probability that more than 30 pairs are defective in a day when 1000 pairs are produced.

32. The number of accesses to a certain web site follows a Poisson process with mean rate of 30 per day.

a. Calculate and compare the values obtained using the probability function and the approximation using the central limit theorem. Coment.

b. What is the probability that the weekly number of accesses (7 days) are between 200 and 220?

c. Determine the probability that in one month (30 days), at the most 5 days have a number of registered accesses bigger than 40.

33. The number of people who heads to a bank branch to acquire a certain art collection, during the six months following the launch, follows a Poisson process with a rate of 10 per month.

a. Determine the minimum number of collections of art to have in stock to be at least 0.9 the probability to satisfy demand in this semester.

b. Knowing that each collection is sold at 3200 Euros, what is the probability that in first quarter revenue from the sale of this product exceeds 140,000 euros?

34. The number of tickets issued for parking at a certain street of Lisbon city follows a Poisson process with an average rate of 50 per day.

a. Get the approximate probability that this number is between 35 and 70, including the extremes.

b. For the five days of the week, what is the probability that this number is between 225 and 275.

36. The number of cargo ships arriving at a port to unload follows a Poisson process with a rate of 2 per day.

a. Knowing that the operating conditions in this port only allows the discharge of up to 4 ships per day, calculate the probability of existence of ships to unload at the end of a day when no service was delayed the previous day.

b. What is the probability that in a month (30 days), at least 70 cargo ships come to port for unloading.

c. What is the probability that in one year (365 days), there are less than 10 days where more than 5 cargo ships arrive at this port per day?

37. Passengers for the Train "Express" with departure to Coimbra at noon start boarding the train from 11:40 onwards. The number of passengers entering the train in the first 15 minutes follows a Poisson process with a mean rate of 2 passengers per minute. In the last 5 minutes between 11:55 to 12:00, the number of passengers who join the train follows a Poisson process with a mean rate of 5 passengers per minute. If the train has a maximum capacity of 60 seats, calculate the probability of being adequate to meet demand.